

Universality in Quantum Hall Systems: Coset Construction of Incompressible States

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Incompressible Quantum Hall fluids (QHF's) can be described in the scaling limit by three-dimensional topological field theories. Thanks to the correspondence between three-dimensional topological field theories and two dimensional *chiral* conformal field theories (CCFT's), we propose to study QHF's from the point of view of CCFT's. We derive consistency conditions and stability criteria for those CCFT's that can be expected to describe a QHF. A general algorithm is presented which uses simple currents to construct interesting examples of such CCFT's. It generalizes the description of QHF's in terms of Quantum Hall lattices. Explicit examples, based on the coset construction, provide candidates for the description of Quantum Hall fluids with Hall conductivity $\sigma_H = \frac{1}{2}(e^2/h)$, $\frac{1}{4}(e^2/h)$, $\frac{3}{5}(e^2/h)$, (e^2/h) ,....

KEY WORDS: Quantum Hall effect; conformal field theory; topological field theory; coset construction; Chern–Simons electrodynamics; simple currents.

1. INTRODUCTION

In this paper we reconsider the fractional quantum Hall effect, using methods from conformal and topological field theory. Besides improving the foundations of a theoretical description of quantum Hall fluids in a dissipation-free (incompressible) state (“incompressible quantum Hall fluid”) in terms of two-dimensional chiral conformal field theory, our main interest is in

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showing that such a description reproduces many important features of incompressible quantum Hall fluids corresponding to Hall conductivities

$$\sigma_H = \frac{1}{2} \frac{e^2}{h}, \frac{1}{4} \frac{e^2}{h}, \frac{e^2}{h}, \dots \quad (1)$$

The quantum Hall effect is observed in two-dimensional (2D) electron gases forming at an interface between a semi-conductor and an insulator when such gases are put into a strong, uniform magnetic field transversal to the plane to which the electrons are confined (by an electric field). Tuning the current through the sample to a fixed value $I = (I_x, I_y)$ and measuring the voltage drops, (V_x, V_y) , in the x - and y -direction, one can determine the longitudinal and Hall resistances from the *Ohm–Hall law*

$$\begin{aligned} V_x &= R_L I_x + R_H I_y \\ V_y &= -R_H I_x + R_L I_y \end{aligned} \quad (2)$$

For a fixed external magnetic field, one can vary the density of electrons in the 2D electron gas by varying the gate voltage, i.e., the electric field perpendicular to the interface to which the electron gas is confined. One then finds that if the electron density belongs to certain intervals (whose width depends on the magnetic field and the strength and density of impurities) the longitudinal resistance vanishes: $R_L = 0$. This is a signal for the *absence of dissipative processes* in the 2D electron gas, which, in turn, can be interpreted as an indication that the gas is in an “*incompressible state*” with a strictly positive mobility gap in the bulk of the sample. Experimentally, one finds that whenever R_L vanishes the Hall conductivity

$$\sigma_H = R_H^{-1}, \quad \text{for } R_L = 0 \quad (3)$$

is a *rational multiple* of e^2/h , where e is the elementary electric charge and h is the Planck constant. This “*quantization*” of the Hall conductivity is extremely precise for well developed plateaux. The phenomena described here are referred to as the *quantum Hall effect*.

The *integer* quantum Hall effect ($\sigma_H = 1(e^2/h)$, $2(e^2/h)$, $3(e^2/h)$, ...) was discovered by von Klitzing, Dorda and Pepper in 1980,⁽¹⁾ the *fractional* quantum Hall effect by Tsui, Störmer and Gossard in 1982.⁽²⁾ Fundamental insights into theoretical explanations of these remarkable effects were soon brought forward by Laughlin.^(3,4) In particular, he discovered a trial wave function accurately encoding properties of the ground state of a quantum Hall fluid in an incompressible state with $\sigma_H = \frac{1}{3}(e^2/h)$, which is now called “Laughlin fluid.” He made the tantalizing observation that, in a Laughlin

fluid, there are quasi-particles of electric charge $\pm e/3$, called “Laughlin vortices.” It was recognized that, besides their fractional electric charge, Laughlin vortices carry one quantum of flux and exhibit fractional (“braid”) statistics; see refs. 5, 6 and references given there. In 1982, Halperin showed that quantum Hall fluids in an incompressible state confined to a finite domain exhibit chiral diamagnetic currents localized near the boundary of the sample, the so called “*edge currents*.”⁽⁷⁾ Halperin’s arguments were based on a direct analysis of the quantum mechanics of 2D non-interacting electron gases under the influence of a transverse magnetic field and of impurities in the bulk of the sample. The important rôle played by edge currents was emphasized in later work by Büttiker,⁽⁸⁾ Beenaker,⁽⁹⁾ and others.⁽¹⁰⁾

In 1989/1990, it was recognized independently by Wen⁽¹¹⁾ and by Fröhlich and Kerler⁽¹²⁾ (see also ref. 13 for a sample of subsequent work) that the diamagnetic edge currents of an arbitrary quantum Hall fluid in an incompressible state are described by quantum mechanical current operators generating a *chiral current* (Kac–Moody) *algebra*. This opened the view towards using methods from (chiral) conformal field theory to analyze incompressible quantum Hall fluids.

It was emphasized in ref. 12 that, in order to study *universal* properties of quantum Hall fluids in an incompressible state, it is convenient to describe such fluids in the so called *scaling limit* in which distances and times are infinitely rescaled. The concept of studying physical systems in the scaling limit is familiar from the theory of *critical phenomena*. It was noticed that, because quantum Hall fluids in an incompressible state exhibit a strictly positive mobility gap, their bulk properties are described, in the scaling limit, by a *topological field theory*.^(12, 14, 15) In particular, the theory describing the quantum-mechanical electric charge- and current density operators in the scaling limit was shown to be an abelian topological *Chern–Simons theory*.^(12, 14) This observation also provided additional insight into the origin of the diamagnetic edge currents: they are necessary to guarantee that the total electric charge is *conserved* in *closed* incompressible quantum Hall fluids; (“*anomaly cancellation*”—it is actually quite fascinating to realize that the diamagnetic edge currents of a quantum Hall fluid in an incompressible state are carried by chiral, quantum-mechanical degrees of freedom violating electromagnetic gauge invariance; this violation of electromagnetic gauge invariance is exactly compensated by one exhibited by the bulk degrees of freedom of the fluid). The conspiracy between edge and bulk degrees of freedom in ensuring conservation of the total electric charge and in cancelling each other’s violations of electromagnetic gauge invariance is an instance of what has recently become known as the “*holographic principle*.”⁽¹⁶⁾ In the example of incompressible

quantum Hall fluids this principle would say that the Hall conductivity σ_H can be measured in experiments involving only edge currents, or in ones involving only bulk currents, or in experiments involving edge *and* bulk currents, and that there is a correspondence between the quasi-particle spectra in the bulk and the quasi-particle spectra of the edge degrees of freedom. In more theoretical terms, the *topological field theory* describing the scaling limit of the bulk of an incompressible quantum Hall fluid is completely determined by a *chiral conformal field theory* describing the edge degrees of freedom of an incompressible quantum Hall fluid with the same Hall conductivity. The connection between three-dimensional topological Chern–Simons theory and the two-dimensional chiral Wess–Zumino–Witten model (Kac–Moody algebra) was discovered by Witten.⁽¹⁷⁾ A fairly general construction of 3D topological field theories from 2D chiral conformal field theories was later described in ref. 18. It will be invoked in this paper.

In general, there is no guarantee that the chiral conformal field theory that determines the topological field theory describing the scaling limit of the *bulk* of an incompressible quantum Hall fluid is identical to the one describing the *edge* degrees of freedom of the fluid; although some of their properties do coincide. The reason is that the structure of the fluid near the boundary of the domain to which it is confined can be quite complicated, exhibiting several distinct, thin layers. Thus, the theory of the edge degrees of freedom is, in general, more complicated, and less universal than the theory of the bulk. This is why we shall emphasize the study of topological field theories describing the scaling limit of the *bulk* of *homogeneous* quantum Hall fluids in incompressible states, i.e., fluids whose Hall conductivity and quasi-particle spectra are everywhere the same in the bulk.

There is no doubt that the idea to analyze universal properties of quantum Hall fluids in incompressible states by considering their scaling limits and using topological field theories to describe them is sound and has turned out to be fruitful. The key question is then how much *concrete information* about incompressible quantum Hall fluids can be gained from such a general and quite abstract approach. Clearly, this approach *cannot* be used to understand for which values of the external control parameters, such as the magnetic field, the electron density, the density and strength of impurities, etc. the ground state of a quantum Hall fluid is incompressible, in the sense that a mobility gap opens and the longitudinal resistance R_L vanishes. An understanding of these problems requires analysis of the microscopic quantum mechanics of 2D interacting electron gases in a transverse magnetic field, and this is hard analytical and/or numerical work; see refs. 6, 19, 20. However, *assuming* that R_L vanishes, methods from 3D topological field theory/2D chiral conformal field theory provide a *key*

to understand which values the Hall conductivity can take, what spectra of quasi-particles may occur in incompressible states and what their quantum numbers are, what fractional electric charges may be measured, whether there may be *several distinct* incompressible states corresponding to the *same* value of the Hall conductivity (“intra-plateaux transitions”), what kind of heat currents may be observed, etc.; see refs. 11, 21–23. The main purpose of our paper is to improve the foundations and generalize the scope of this approach. The pay-off will be to provide very plausible descriptions of incompressible quantum Hall fluids with Hall conductivities $\sigma_H = \frac{1}{2}(e^2/h)$, $\frac{1}{4}(e^2/h)$, $e^2/h, \dots$. Accurate descriptions of incompressible quantum Hall fluids with $\sigma_H = (N/2pN + 1)(e^2/h)$, $p = 1, 2, 3$, $N = 1, 2, \dots, 8, \dots$ have been presented, within the general approach described above, in refs. 21–23. They have features fairly closely related to, e.g., Jain’s description of these fluids.⁽²⁴⁾ But the approach in refs. 21–23 was not quite general enough to provide a plausible description of, e.g., an incompressible quantum Hall fluid with Hall conductivity $\sigma_H = \frac{1}{2}(e^2/h)$, which is observed in *double-layer systems*. In the scaling limit, the theoretical description of such a fluid can be expected to have two important *symmetries*, an $SU(2)$ -layer symmetry and the $SU(2)$ of quantum mechanical spin; (see, e.g., ref. 23). It can happen, however, that the diagonal $SU(2)$ -subgroup of the symmetry group $SU(2)_{\text{layer}} \times SU(2)_{\text{spin}}$ is *not* a global symmetry of the fluid—it is “gauged.” [Adding an electron with “spin up” in one layer may turn out to be “gauge-equivalent” to adding an electron with “spin down” in the other layer—roughly speaking.] This possibility appears to be realized in the “*Pfaffian state*” proposed by Moore and Read.^(25, 26) One concrete goal of this paper is to generalize the approach of refs. 21–23 by “gauging” subgroups of symmetry groups of incompressible quantum Hall fluids. In several physically interesting examples, this construction, usually referred to as the *coset construction*,^(27, 28) does *not* change the value of the Hall conductivity; but it changes the quasi-particle spectrum (by identifying certain quasi-particles that were formerly distinct and by turning some formerly elementary quasi-particles into composite quasi-particles); see Appendix B for a more precise treatment. It leads to a natural theoretical interpretation of various trial ground-state wave functions, such as the one in ref. 25. Some examples of our general approach have been described in ref. 29, but with an emphasis on properties of edge states rather than of bulk states.

Historically, the problem of the *quantization* of the Hall conductivity of quantum Hall fluids in incompressible states and of elucidating other physical properties of such fluids has of course been studied since the discovery of the quantum Hall effect. A highly original theory of quantum transport has been developed, for this and other purposes, by Thouless and

coworkers⁽³⁰⁾ and their followers⁽³¹⁾; see also ref. 32 and references given there. In this approach the Hall conductivity is identified with a Chern number. An alternative approach, identifying the Hall conductivity with an index, was proposed in refs. 33–35. An approach towards understanding the quantization of the Hall conductivity in terms of edge states has been described in ref. 36; see also refs. 3, 37–39. In all these approaches, the 2D electron gas is treated as *noninteracting*, which severely limits their scope. The observation that one can come up with *general predictions* of the possible values of the Hall conductivity, of the spectrum of quasi-particles and of their quantum numbers, such as their (generally fractional) electric charges, of incompressible quantum Hall fluids by merely assuming that the longitudinal resistance R_L vanishes and then using methods from topological field theory/chiral conformal field theory was made in refs. 12, 14, 15. It is based on a fundamental connection between the electric charge of a cluster of quasi-particles and its quantum statistics, which was first described in ref. 12. Besides an analysis of concrete examples, a general implementation and improved presentation of these ideas is among the main purposes of our paper.

We conclude this introduction with a brief outline of the contents of this paper. In Section 2, we first recall the laws of electrodynamics of quantum Hall fluids in an incompressible state ($R_L = 0$), starting from the *Ohm–Hall law*. We then show that these laws alone imply the existence of (“anomalous”) chiral edge currents generating a chiral current (Kac–Moody) algebra. Subsequently, the connection between the laws of electrodynamics of incompressible quantum Hall fluids and Chern–Simons theory and between the latter and the “anomalous nature” of the edge currents is briefly recalled. We then review how a description of incompressible quantum Hall fluids in the scaling limit leads one to consider 3D topological field theories generalizing the electromagnetic Chern–Simons theory. We argue that those topological field theories that are relevant in a theoretical description of incompressible quantum Hall fluids in the scaling limit can be constructed from 2D chiral conformal field theories, and we sketch some basic aspects of this construction; (for details see ref. 18). Of course, chiral conformal field theories also appear in the description of the edge degrees of freedom of incompressible quantum Hall fluids with the same Hall conductivity.

In Section 3, we formulate fundamental conditions, “*consistency conditions*,” singling out those topological field theories/chiral conformal field theories that can, in principle, appear in the description of the scaling limit of an incompressible quantum Hall fluid. We comment on the role played by an intriguing mathematical property of such theories, “modular covariance,” in determining the full spectrum of quasi-particles of an

incompressible quantum Hall fluid (clarifying, perhaps, some misconceptions that have appeared in the literature). We then recall some phenomenological criteria enabling one to assess the *stability* of an incompressible quantum Hall fluid described by a given topological field theory. They represent an elaboration on criteria proposed in refs. 21–23.

In Section 4, we recall the construction of a *special class of topological field theories* relevant for a theoretical description of incompressible quantum Hall fluids that was identified and studied in refs. 21–23. Theories in this special class are in a one-to-one correspondence with certain (odd) integral lattices called “*quantum Hall lattices*.” The main properties of quantum Hall lattices are recalled, and the consistency conditions and stability criteria formulated in Section 3 are used to derive some constraints that must be imposed on quantum Hall lattices.

In Section 5, a more general class of topological field theories/chiral conformal field theories expected to be relevant for a description of incompressible quantum Hall fluids is identified. It is explained how to calculate the Hall conductivity in such theories and why it is necessarily a *rational multiple* of e^2/h . The value of the smallest fractional electric charge that can appear as the charge of a quasi-particle of a quantum Hall fluid described by such a theory is calculated. It is then indicated how examples of topological field theories from the class of theories described at the beginning of Section 5 can be constructed from the theories described in Section 4, which are characterized by quantum Hall lattices, by making use of the so called *coset construction*.^(27, 28, 40)

In Section 6, we discuss *concrete examples* of topological field theories describing interesting quantum Hall fluids in incompressible states. These examples are based on “Virasoro minimal models,” such as the 2D chiral Ising model, and “simple current extensions” thereof. Among our examples are theories describing incompressible quantum Hall fluids with Hall conductivity $\sigma_H = \frac{1}{2}(e^2/h)$, $\frac{1}{4}(e^2/h)$ and e^2/h . Perhaps, the most interesting example is a theory with $\sigma_H = \frac{1}{2}(e^2/h)$, which is related to the 2D chiral Ising model. It predicts properties of an incompressible quantum Hall state at $\sigma_H = \frac{1}{2}(e^2/h)$ compatible with those predicted by the “Pfaffian state” of Moore and Read.⁽²⁵⁾ Our results on specific examples of incompressible quantum Hall fluids are summarized in four tables.

In Appendix A, “modular covariance” of the theories described in Sections 5 and 6 is discussed, and in Appendix B some basic facts concerning the “coset construction” are recalled.

We dedicate this paper to the memory of Quin Luttinger, who made fundamental contributions to diverse fields of theoretical physics ranging from relativistic QED over condensed-matter physics to mathematical physics. With his open mind, his charming, friendly personality, his curiosity

and his original thinking he inspired colleagues in every field to which he turned his interest. One of his far-reaching contributions concerned the theory of one-dimensional electron gases, systems now known as *Luttinger liquids*. The chiral degrees of freedom in an incompressible quantum Hall fluid represent an example of a one-dimensional gas of electrons or holes. It is sometimes called a “chiral Luttinger liquid.” It is plausible that Luttinger would have found these systems interesting.

2. GENERAL FEATURES OF QUANTUM HALL FLUIDS

A quantum Hall fluid (QHF) is a two-dimensional interacting electron gas in a compensating uniformly charged background, subject to a magnetic field transversal to the confinement plane. Among the experimental control parameters is the filling factor defined as

$$\nu = \frac{n^{(0)}}{B^{(0)}} \frac{hc}{e} \quad (4)$$

where $n^{(0)}$ is the electron density, $B^{(0)}$ denotes the uniform transverse magnetic field, and hc/e is the flux quantum.

Electric transport properties of a QHF in a small electric field at low frequency are described by the relation between the electric field parallel to the plane of the sample and the (expectation value of the quantum mechanical) electric current,

$$\vec{J} = \begin{pmatrix} \sigma_L & \sigma_H \\ -\sigma_H & \sigma_L \end{pmatrix} \vec{E} \quad (5)$$

(the Ohm–Hall law, compare (2)), where σ_L is the longitudinal conductivity and σ_H the transverse or Hall conductivity.

Experimentally, it is observed that, in certain intervals of the filling factor, the longitudinal conductivity vanishes,⁽²⁾ a sign that dissipative processes are absent in the fluid. Moreover, it is observed that, on such intervals, the Hall conductivity is a rational multiple of e^2/h . For reasons that will become clear later, we call a QHF with these properties “incompressible.”

2.1. Electrodynamics of Incompressible Quantum Hall Fluids

The basic equations of the electrodynamics of an incompressible QHF can be derived as follows (see ref. 12): In $(2+1)$ dimensions, the electromagnetic field tensor is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & B \\ -E_2 & -B & 0 \end{pmatrix} \quad (6)$$

where E_1 , E_2 are the components of the electric field in the sample plane and B is a perturbation of the transverse background magnetic field, such that $B^{(\text{total})} = B^{(0)} + B$. From the $(2+1)$ -dimensional homogeneous Maxwell equations (Faraday's induction law),

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \quad (7)$$

the continuity equation for the electric current density (conservation of electric charge),

$$\partial_\mu J^\mu = 0 \quad (8)$$

and from the transport equation (5) for $\sigma_L = 0$, i.e.,

$$\vec{J} = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix} \vec{E} \quad (9)$$

it follows that

$$J^0 = \sigma_H B \quad (10)$$

Equations (9) and (10) can be combined to the equation

$$J^\mu = \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda} \quad (11)$$

of Chern–Simons electrodynamics.⁽⁴¹⁾ It describes the response of an incompressible QHF to an external electromagnetic field. It is compatible with the continuity equation (8) iff σ_H is constant.

We may take into account the finite extension of the sample confined to a region $\Omega = D \times \mathbf{R}$ of space-time, by setting the Hall conductivity σ_H to a nonzero, constant value on Ω and to zero outside, i.e.,

$$\sigma_H(\cdot) = \sigma_H \chi_\Omega(\cdot) \quad (12)$$

where $\chi_\Omega(\cdot)$ is the characteristic function of the space-time region Ω . The divergence of the electric current density (11) is then different from zero on the boundary of the sample:

$$\partial_\mu J^\mu = \sigma_H \varepsilon^{\mu\nu\lambda} \partial_\mu \chi_\Omega F_{\nu\lambda} \quad (13)$$

Since conservation of electric charge in closed systems is a law of nature, there must be an electric current J_{edge} localized at the boundary $\partial\Omega$ of the sample, with the property that the *total* electric current

$$J_{\text{total}}^{\mu} = J^{\mu} + J_{\text{edge}}^{\mu} \quad (14)$$

is divergencefree. The edge current, J_{edge}^{μ} , has the form $J_{\text{edge}}^{\mu} = j^{\mu} \delta_{\partial\Omega}$, where j^{μ} is a current density on the boundary $\partial\Omega$ whose component normal to $\partial\Omega$ must vanish. Equation (13) then implies that

$$\partial_{\alpha} j^{\alpha} = \frac{1}{2} \sigma_H \varepsilon^{\alpha\beta} F_{\alpha\beta} \quad (15)$$

Here, the indices α, β refer to coordinates for the $(1+1)$ -dimensional boundary $\partial\Omega$.

Equation (15) expresses the $(1+1)$ -dimensional (abelian) chiral anomaly, see, e.g., ref. 42, and tells us that there are chiral (and hence gapless) degrees of freedom localized on the boundary, which are coupled to the electromagnetic gauge field in such a way that the electric current they carry obeys the anomaly equation (15). Quantum mechanically, this current is described by a $\hat{u}(1)$ -current algebra, with an anomalous commutator proportional to the Hall conductivity:

$$[j_m, j_n] = \delta_{m+n, 0} \sigma_H \quad (16)$$

where j_n is the n th Fourier component of j ; see refs. 12, 11, and 23.

2.2. Topological Field Theory and Incompressible Quantum Hall Fluids

Next, we return to studying the physics of the bulk of the sample. The absence of dissipation in the transport of electric charge (9) can be explained by the existence of a gap in the energy spectrum between the ground state energy of the QHF and the energies of excited (extended) bulk states.^(12, 11) This explains the term “incompressible”: it is not possible to add an additional electron to the fluid, or to extract one from the fluid, by paying only an arbitrarily small energy. An important consequence of incompressibility is that the total electric charge is a good quantum number to label different sectors of physical states at zero temperature.

We are interested in the physics in the scaling limit of an incompressible QHF, i.e., in the limit in which short-distance- and high-frequency properties become invisible. This limit is defined as follows.^(12, 23) For an arbitrary disc D , consider the family of fluids confined to discs of different sizes $D_{\theta} = \{x \mid \Phi^{-1}x \in D\}$. The Green functions in the scaling limit can be

constructed from the Green functions of the systems confined to D_θ as follows:

$$G_{\lambda_1, \dots, \lambda_n}^D(x_1, \dots, x_n) = \lim_{\theta \rightarrow \infty} \theta^\sigma \langle T[\phi_{\lambda_1}(\theta x_1) \cdots \phi_{\lambda_n}(\theta x_n)] \rangle_{(D_\theta)} \quad (17)$$

where the ϕ_λ are fields of the theory that describes the fluid, and the exponent $\sigma(\lambda_1, \dots, \lambda_n)$ on the right hand side takes into account the scaling dimensions of the fields appearing in the time-ordered product. Thus, in the scaling limit, the fluid is considered to be confined to a standard disc D . Our goal is to describe the space of physical state vectors of an incompressible QHF in the scaling limit.

The presence of a positive energy gap implies that, in this limit, the theory describing an incompressible QHF is a “topological field theory,” whose excitations are *static pointlike sources* localized in the bulk and labelled by quantum numbers, such as electric charge, or, perhaps, spin. More formally, the pointlike sources are marked by elements λ in a set A , characteristic of the fluid, which generates a *fusion ring*, a term coming from representation theory that will be defined below.

Equation (11), which relates the ground state expectation value of the electric current to the external electromagnetic field, is an expression of the fact that the bulk theory *in the scaling limit* is *topological*. To see this, consider the topological (metric-independent) $(2+1)$ -dimensional Chern–Simons (CS) action

$$CS_3[A] = \int d^3x \sigma_H \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \quad (18)$$

This is the effective action of the bulk degrees of freedom in an external electromagnetic field with vector potential A_μ . In fact, it gives the correct equation for the electric current density:

$$J^\mu(x) = \frac{\delta}{\delta A_\mu(x)} CS_3[A] = \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda} \quad (19)$$

For the current-current correlation function, the CS effective action yields

$$\langle J^\mu(x) J^\nu(y) \rangle = \frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(y)} CS_3[A] = \sigma_H \varepsilon^{\mu\nu\lambda} \partial_\lambda \delta(x-y) \quad (20)$$

which is the unique expression for the leading term in the scaling limit for a system with broken parity, as can be deduced from dimensional analysis.⁽²³⁾

The CS action is not invariant under gauge transformations that do not vanish on the boundary $\partial\Omega$ of the sample space-time. But its gauge variation is exactly compensated by the variation of the effective action for the coupling of the electromagnetic gauge field to the boundary degrees of freedom described by the $\hat{u}(1)$ -current algebra, i.e., the two-dimensional anomalous chiral action.⁽⁴²⁾

2.3. Two-Dimensional Chiral Conformal Field Theory and Three-Dimensional Topological Field Theory

In the last subsections we have argued that the Green functions of the electromagnetic current density of an incompressible QHF are described, in the scaling limit, by a “topological field theory;” see (19) and (20). More generally, the entire physics of an incompressible QHF can be encoded, in the scaling limit, into a “topological field theory” defined over the three-dimensional space-time of the sample. The purpose of this section is to recall what one means by a “three-dimensional topological field theory,” and to explain the connection between some of these theories and “two-dimensional chiral conformal field theories.”

Generally speaking, a *three-dimensional topological field theory* (3D TFT) associates a topological invariant to every three-dimensional manifold without boundary. However, a finite Hall-sample has a boundary. Its space-time is therefore described by a three-dimensional manifold *with* boundary. Thus, it must be possible to define those 3D TFT’s that describe the physics of incompressible QHF’s in the scaling limit on three-dimensional manifolds with boundary, provided the boundary is given a suitable geometric structure. Furthermore, it is reasonable to expect that some kind of “holographic principle” is valid: the TFT on a three-dimensional manifold, Ω , with boundary $\partial\Omega$, denoted by Σ , should be unambiguously determined by a two-dimensional field theory defined over Σ .

In the bulk of an incompressible QHF there may be static sources labelled by some quantum numbers in a fusion ring. In rescaled space-time, such sources trace out worldlines. At certain times, two sources may collide, i.e., their worldlines may be fused into a single worldline, or a source may split into two distinct sources. Thus, the worldlines of sources in the bulk of an incompressible QHF can be viewed as the lines of “Feynman diagrams” with trivalent vertices,⁴ each oriented line in the diagram carrying the quantum numbers of the source it represents; see Fig. 1. Since the effective field theory describing an incompressible QHF in the scaling limit is topological, it is only the *topology* of the Feynman diagram

⁴ Higher vertices can be reduced to trivalent vertices by repeated fusing.

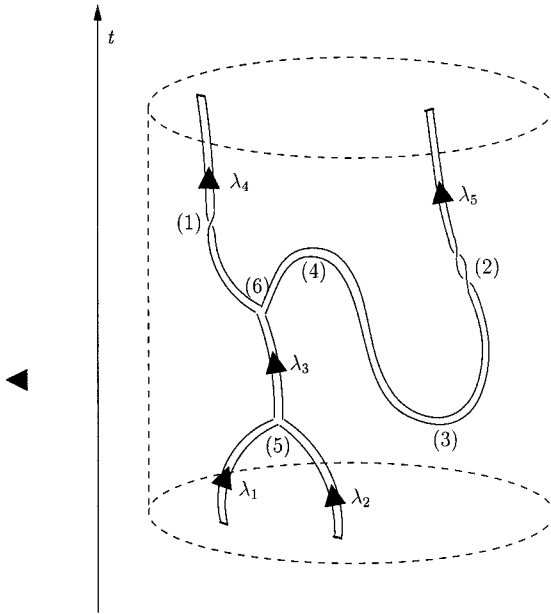


Fig. 1. A Feynman diagram or ribbon graph for some sources. One can recognize (1) a self-twist, (2) a double self-twist, (3) a pair production, (4) a pair annihilation, (5) two sources that fuse into one, and (6) a source that splits in two.

representing the worldlines of the sources that matters. It is important to realize that the Feynman diagrams are “framed” diagrams, i.e., along each line in the diagram a field of vectors (perpendicular to the tangent vector at each point of the line) is defined which enables us to keep track of the “self-twist” of the line. Such “self-twists” appear as Aharonov–Bohm type phase factors in the quantum mechanical transition amplitudes of an incompressible QHF. In fact, the value of a Feynman diagram representing the worldlines of bulk sources in an incompressible QHF is nothing but a “generalized Aharonov–Bohm phase” depending only on the *topology* of the diagram, including the self-twists of its lines, i.e., depending only on the “ribbon graph” traced out by the Feynman diagram.

Thus, in order to describe incompressible QHF’s in the scaling limit, we are looking for 3D TFT’s which can be defined on manifolds with boundary in which some oriented ribbon graph is inscribed, whose lines are decorated with quantum numbers from a fusion ring. Moreover, these TFT’s should be determined by field theories on the *boundary* of the three manifold in such a way that the “holographic principle” is satisfied.

It turns out that every two-dimensional chiral conformal field theory (2D CCFT) determines a 3D TFT with all the properties required above. In fact, it is reasonable to expect that every 3D TFT with the above properties can be derived from a 2D CCFT. We assume this as a justification to constrain the class of 3D TFT's used to describe the physics in the bulk of an incompressible quantum Hall fluid in the scaling limit to those derived from 2D CCFT's. [These CCFT's are, however, not unique: the WZW-theories based on $so(n)$ and $so(n+16)$ at level one, e.g., provide identical 3D TFT's.] Thus, we recall what is meant by a two-dimensional chiral conformal field theory.

A *two-dimensional chiral conformal field theory* is a quantum field theory defined over a cylindrical space-time $\mathbf{R} \times S_R^1$ of radius R , with coordinates (t, φ) . The quantum-mechanical degrees of freedom are *chiral*, which means that the dynamical modes of such a theory are purely left- or purely right-moving.⁵ Put differently, all the fields of a 2D CCFT only depend on *one* light-cone coordinate, say $u_- = vt - R\varphi$, with v the propagation velocity of the modes. Among these fields, we consider all the *local* ones. With the help of the operator product expansion one shows that the local chiral fields form an algebra of operators (operator-valued distributions) on the Hilbert-space of physical states. This algebra is called the *chiral algebra* of the theory and is denoted by \mathcal{A} . Among the fields generating \mathcal{A} , there is the energy-momentum tensor of the theory. Because the theory is assumed to be chiral, its energy-momentum tensor is *traceless*. Its Fourier components (with respect to the coordinate φ), L_n , then satisfy the commutation relations of the *Virasoro algebra*, which is related to a central extension of the Lie algebra of infinitesimal conformal transformations—hence the term “conformal” field theory.

A 2D CCFT with chiral algebra \mathcal{A} can be reconstructed from the unitary representations of \mathcal{A} .⁽¹⁸⁾ We need to recall the notions of conformal weight and fusion rules which come from representation theory.

Let λ be a unitary representation of \mathcal{A} , and let \mathcal{H}_λ denote the corresponding representation space, which is a Hilbert space. To each such representation one assigns a nonnegative number, A_λ , called the *conformal weight* of the representation. It is defined as the minimal eigenvalue of the zero-mode operator, L_0 , of the energy-momentum tensor,

$$A_\lambda = \inf \{ \langle v_\lambda, L_0 v_\lambda \rangle \mid v_\lambda \in \mathcal{H}, \|v_\lambda\| = 1 \} \quad (21)$$

⁵ The choice of left moving modes or right moving modes endows the boundary with an orientation. In fact, CCFT can be naturally considered also on closed, oriented surfaces of higher genus. In the description of a QHF, such surfaces do not appear in physically meaningful situations.

In a consistent theory there is always an irreducible *vacuum representation*, ω , characterized by the vanishing of the conformal weight, $A_\omega = 0$.

Given two representations, λ and μ , one can define their *fusion*, namely a *tensor product representation*, $\lambda * \mu$, which is again a unitary representation of \mathcal{A} . A chiral algebra is called *rational* iff the number of inequivalent, irreducible unitary representations is finite. Let us denote by A the set of such representations. For a rational chiral algebra, the tensor product of two representations can be decomposed into a direct sum of irreducible unitary representations. Thus, the set of unitary irreducible representations of a rational chiral algebra, furnished with the tensor product, has the structure of a commutative, associative ring. For λ_1, λ_2 and λ_3 in A , let $N_{\lambda_1, \lambda_2}^{\lambda_3}$ denote the multiplicity of λ_3 as a subrepresentation in the tensor product $\lambda_1 * \lambda_2$. The multiplicities $N_{\lambda_1, \lambda_2}^{\lambda_3}$ are the structure constants of the ring and are called *fusion rules*; for a rational chiral algebra, they are finite non-negative integers. The vacuum representation, ω , plays the rôle of the unit for the tensor product, i.e., $\lambda * \omega = \omega * \lambda = \lambda$. To every irreducible representation λ there corresponds a contragradient (or conjugate) representation $\bar{\lambda}$ with the property that $\lambda * \bar{\lambda}$ contains the vacuum representation ω exactly once as a subrepresentation.

Given a number n of irreducible unitary representations, $\lambda_1, \dots, \lambda_n$, we define the linear space of *conformal blocks* as the space of invariant tensors, i.e., of invariant linear functionals, on the representation space of the tensor-product representation $\lambda_1 * \dots * \lambda_n$. It actually turns out (see ref. 43) that the tensor product representation $\lambda_1 * \dots * \lambda_n$ depends on n complex parameters z_1, \dots, z_n , which can be considered as coordinates of pairwise different points of the complex plane to which the cylinder can be mapped. For this reason, the space

$$V_{S^2}(z_1, \lambda_1, \dots, z_n, \lambda_n) \tag{22}$$

of conformal blocks depends on the complex parameters z_1, \dots, z_n . Its dimension is given by

$$\mathcal{N}_{\lambda_1, \dots, \lambda_n} = \sum_{\mu_1, \dots, \mu_{n-3}} N_{\lambda_1 \lambda_2}^{\mu_1} N_{\mu_1 \lambda_3}^{\mu_2} \dots N_{\mu_{n-3} \lambda_{n-1}}^{\bar{\lambda}_n} \tag{23}$$

and does not depend on the parameters z_1, \dots, z_n .

Next, we explain in which way 2d CCFT's arise in the description of incompressible QHF's in the scaling limit. We wish to describe the physical state space describing the scaling limit of an incompressible QHF confined to a disc D , which for our purposes can be viewed as a punctured two-dimensional sphere S^2 , the boundary being mapped to $z = \infty$. It turns out that this state space can be identified with the space of conformal blocks

of some CCFT! Let \mathcal{A} be the chiral algebra characterizing a CCFT. The representations of \mathcal{A} are used as the quantum numbers labelling the static sources in the bulk of the QHF. The boundary conditions can be described by vectors in a representation space of the chiral algebra. Fixing a boundary condition $v_\lambda \in \mathcal{H}_\lambda$, and inserting static sources labelled by quantum numbers $\lambda_1, \dots, \lambda_n$ corresponding to representations of \mathcal{A} at points z_1, \dots, z_n in the disc D , the *space of physical states* of the QHF is identified with the space of conformal blocks⁽⁴⁴⁾

$$V_{S^2}(z = \infty, \bar{\lambda}, z_1, \lambda_1, \dots, z_n, \lambda_n)[v_\lambda] \quad (24)$$

with the vector v_λ inserted in the first argument (corresponding to the point $z = \infty$). We denote this space by $\mathcal{H}_{z, \bar{\lambda}}[v_\lambda]$. In order to select a specific vector in $\mathcal{H}_{z, \bar{\lambda}}[v_\lambda]$ and, in particular, to fix the generalized Aharonov–Bohm phases, we consider an adiabatic evolution of sources in the QHF described by a ribbon graph, \mathcal{G} , with $n + 1$ external lines decorated by the representations $\lambda, \lambda_1, \dots, \lambda_n$ ending at the points ∞, z_1, \dots, z_n respectively; see Fig. 2. To each vertex of the ribbon graph, one associates a coupling, which is an element of a linear space whose dimension is given by the corresponding fusion rule (e.g., at the vertex V_1 of Fig. 2, the dimension is

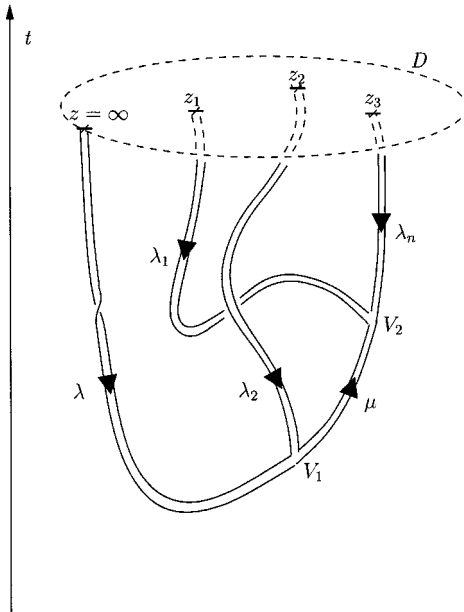


Fig. 2. A ribbon graph for a state in $\mathcal{H}_{z, \bar{\lambda}}[v_\lambda]$.

given by $N_{\lambda\bar{\lambda}}^{\mu}$). It is well known that these data precisely specify a conformal block

$$|\psi_{\mathcal{G}}\rangle \in \mathcal{H}_{\bar{z}, \bar{\lambda}}(v_{\lambda}) \quad (25)$$

It remains to describe the scalar product,

$$\langle \psi_{\mathcal{G}} | \psi_{\mathcal{G}'} \rangle \quad (26)$$

of two vectors $|\psi_{\mathcal{G}}\rangle \in \mathcal{H}_{\bar{z}, \bar{\lambda}}(v_{\lambda})$ and $|\psi_{\mathcal{G}'}\rangle \in \mathcal{H}_{\bar{z}, \bar{\lambda}}(w_{\lambda})$. This scalar product is given by

$$I_{\bar{\mathcal{G}} \amalg \mathcal{G}'} \cdot \langle v_{\lambda}, w_{\lambda} \rangle \quad (27)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in the representation space \mathcal{H}_{λ} , and $I_{\bar{\mathcal{G}} \amalg \mathcal{G}'}$ is the invariant the three-dimensional topological field theory assigns to the ribbon graph obtained by gluing the reflected version $\bar{\mathcal{G}}$ of the ribbon graph \mathcal{G} to the ribbon graph \mathcal{G}' at the end points of the external lines, and $\bar{\mathcal{G}}$ is the ribbon graph obtained from \mathcal{G} by reversing the orientation of the lines of \mathcal{G} ; see Fig. 3. The invariant $I_{\bar{\mathcal{G}} \amalg \mathcal{G}'}$ is a generalized Aharonov–Bohm phase and can be calculated from the data of the underlying 2d CCFT (the fusion rules, the fusing and braiding matrices,...); see ref. 18. The three-dimensional theory with transition amplitudes given by (26), (27) is called a *topological* field theory, because these transition amplitudes *only* depend on the *topology* of $\bar{\mathcal{G}} \amalg \mathcal{G}'$, and not on the precise way in which $\bar{\mathcal{G}} \amalg \mathcal{G}'$ is embedded into three-dimensional space-time.

The explicit expressions for the vectors $|\psi_{\mathcal{G}}\rangle$, in particular their dependence on the insertion points z_1, \dots, z_n , may remind one of generalized Laughlin ansatz wave functions.^(4,6) However, this resemblance is largely

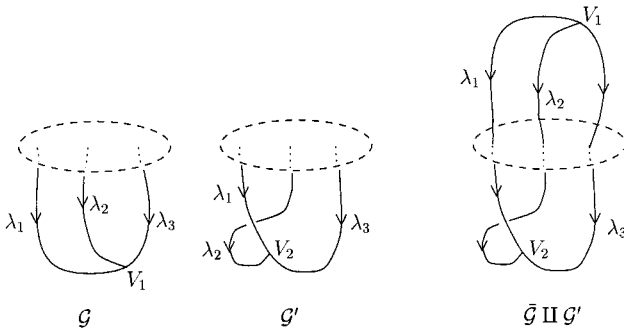


Fig. 3. The ribbon graph $\bar{\mathcal{G}} \amalg \mathcal{G}'$ obtained from the ribbon graphs of \mathcal{G} and \mathcal{G}' .

accidental (and gauge-dependent)! Conformal blocks, both of unitary and non-unitary conformal field theories, can provide a useful description of ‘special functions’ and have been used for this purpose also in other situations, e.g., for the prepotential in Seiberg–Witten models⁽⁴⁵⁾ and for the description of wave functions for the BCS Hamiltonian.⁽⁴⁶⁾ No connection of this kind will be invoked in this paper!

3. CONDITIONS ON A CHIRAL CONFORMAL FIELD THEORY DESCRIBING A QUANTUM HALL FLUID

Thanks to the correspondence between a class of 3D TFT’s and 2D CCFT’s discussed in Section 2, we can study theories describing incompressible QHF from the point of view of CCFT, instead of TFT. In the following, we describe properties that characterize a CCFT whose corresponding TFT can be expected to be relevant in describing an incompressible QHF; we will denote these CCFT’s as quantum Hall CCFT’s.

3.1. Consistency Conditions

To define a CCFT, we must specify a chiral algebra \mathcal{A} and a set A of *unitary* irreducible representations containing a unique vacuum representation ω (which is characterized by the vanishing of the conformal weight $\Delta_\omega = 0$).

As observed in Section 2, the chiral algebra of a Quantum Hall CCFT must contain a $\hat{u}(1)$ -current algebra. This means that the static pointlike excitations of the bulk carry an additive quantum number, which is interpreted as the electric charge of the source. For simplicity, and because we are not attempting a complete classification, we assume that the chiral algebra is a direct product

$$\mathcal{A} = \mathcal{C} \otimes \hat{u}(1) \quad (28)$$

where \mathcal{C} is an *electrically neutral* chiral algebra. Let Π denote the set of unitary irreducible representations of \mathcal{C} , which is closed under fusion. We furthermore assume that \mathcal{C} is *rational*, i.e., that Π is finite. Physically, this means that, for a fixed electric charge, there are only finitely many different static pointlike sources, also called quasi-particles, that carry that charge.

Before going on, let us recall some facts concerning the $\hat{u}(1)$ -theory that are needed later. The unitary irreducible representations are labelled by ‘charges’, i.e., by real numbers r ; the conformal weights are given by $\Delta_r = r^2/2$, and the corresponding fields are vertex operators, $:e^{ir\phi}$, which

are Wick-ordered exponentials of a massless, chiral free field ϕ . The electric current j is expressed in terms of ϕ by $j = \sqrt{\sigma_H} \partial\phi$; it follows that the electric charge, \mathbf{q} , of a representation is $\mathbf{q}_r = \sqrt{\sigma_H} r$.

The following consistency conditions, (C1) through (C5), for the pair (\mathcal{A}, A) reflect physical principles and pragmatic considerations that have proven to be successful in a previous classification of incompressible QHF based on abelian current algebra; see ref. 23.

(C1) Physical Representations. Unitary representations of \mathcal{A} are constructed as tensor products of representations of \mathcal{C} and of $\hat{u}(1)$. Let us denote by A the set of physically realized representations. Since \mathcal{A} is a direct product, we have

$$A \subseteq \Pi \times \mathbf{R} \quad (29)$$

which means that representations of \mathcal{A} are of the form $l = (\pi, r)$, with $\pi \in \Pi$ and r a real number. If two pointlike excitations meet at the same point in the bulk, they generate another pointlike excitation of the fluid. This is a way to express the requirement that the set, A , of representations be closed under fusion.

(C2) Existence of One-Electron States. Among the physically realizable representations there should be (at least) one representation e with electric charge -1 . A pointlike source labelled by e represents an electron that has been inserted somewhere in the bulk. We denote the corresponding representation by $e = (\varepsilon, r_e)$. To say that e has electric charge -1 means that

$$\sqrt{\sigma_H} r_e = -1 \quad (30)$$

We thus require that there be a nonempty family of representations $A_e = \{e_a \mid a = 1, \dots, v_e\}$ in A satisfying (30) which we call (one-) electron representations.

We then define a family A_m in A of multi-electron representations as the representations obtained by multiple fusing of representations in A_e . This means that a pointlike source labelled by such a representation is obtained by letting several electron sources coalesce in one single point, generating a multi-electron cluster. The electric charge of representations in A_m can be determined by making use of the fact that the electric charge is an *additive* quantum number under fusion.

(C3) Charge and Statistics. Consider the state corresponding to a single pointlike source λ in the bulk. If the sample is rotated by 2π with

respect to an axis perpendicular to the sample plane, then the resulting vector differs from the initial one by a phase $e^{2\pi i A_\lambda}$, where A_λ is the conformal weight of the representation λ . If A_λ is an integer, then λ is said to obey Bose-statistics; if A_λ is half-integer, then λ is said to obey Fermi-statistics; if $A_\lambda \neq 0 \pmod{\frac{1}{2}}$, then λ obeys fractional statistics.

We require that those excitations of the incompressible QHF that have been identified with multi-electron pointlike sources obey Fermi/Bose-statistics depending on whether they contain an odd or an even number of electrons, respectively; i.e., we require that, for the multi-electron representations $m \in A_m$, the following *charge-statistics connection* holds:

$$\begin{aligned} \mathbf{q}_m = 0 \pmod{2} &\Rightarrow A_m = 0 \pmod{1} && \text{(Bose statistic)} \\ \mathbf{q}_m = 1 \pmod{2} &\Rightarrow A_m = \frac{1}{2} \pmod{1} && \text{(Fermi statistic)} \end{aligned} \quad (31)$$

Here, \mathbf{q}_m denotes the electric charge of the multi-electron representation m .

(C4) Relative Locality. States with a multi-electron cluster in the bulk should be single valued functions of the position of that cluster. This requirement, which goes under the name of *relative locality*, means that when a multi-electron pointlike source is moved along a closed path in the bulk, possibly winding around other static pointlike sources, then the final state vector is the same as the initial state vector. It turns out that this requirement can be expressed in terms of fusion rules and conformal weights as follows:

$$\text{For all } \lambda, \lambda' \in A, m \in A_m, \text{ with } N_{\lambda m}^{\lambda'} \neq 0, \quad A_m + A_\lambda - A_{\lambda'} = 0 \pmod{1} \quad (32)$$

(C5) Charge and Spin. If spin is a nontrivial quantum number, then, in multi-electron states, it should be determined by the spins of the electrons. Spin labels the representations of an $\widehat{su}(2)_k$ current algebra in the electrically neutral factor \mathcal{C} of \mathcal{A} . If we identify a subalgebra $\widehat{su}(2)_k \in \mathcal{C}$ as describing spin, then we require the multi-electron representations $m \in A_m$ to obey a *spin and charge connection*:

$$\begin{aligned} \mathbf{q}_m = 0 \pmod{2} &\Rightarrow s_m = 0 \pmod{1} \\ \mathbf{q}_m = 1 \pmod{2} &\Rightarrow s_m = \frac{1}{2} \pmod{1} \end{aligned} \quad (33)$$

Here, s_m denotes the $\widehat{su}(2)$ -spin of the representation m .

At this point, an important remark should be made: From the assumption that \mathcal{C} be rational and from condition (C3) it follows that the

Hall conductivity is a *rational number*. This can be seen as follows: The conformal weight of an electron representation is given by

$$\Delta_e = \frac{r_e^2}{2} + \Delta_\varepsilon = \frac{1}{2\sigma_H} + \Delta_\varepsilon \quad (34)$$

and, by (C3), it must be half-integer, say $\frac{1}{2} + j$ with j a positive integer. It is known that, for a rational CCFT, the conformal weights are rational numbers. Thus, Δ_ε in (34) is a rational number. It then follows that the Hall conductivity

$$\sigma_H = \frac{1}{1 + 2j - 2\Delta_\varepsilon} \quad (35)$$

is *rational*.

Next, we define a simple, but useful transformation, called the *shift map* (compare ref. 23), that maps a Quantum Hall CCFT to another Quantum Hall CCFT modifying only the electrically charged part of the theory: the electric current is transformed as

$$j' = \sqrt{\frac{\sigma_H}{1 + 2p\sigma_H}} j \quad (36)$$

where the shift parameter p is a positive integer. The Hall conductivity is transformed as

$$\sigma'_H = \frac{1}{1 + 2p\sigma_H} \sigma_H \quad (37)$$

The $\hat{u}(1)$ -label of the representations is scaled in such a way that the electron representations continue to have charge equal to 1: $r'_e = r_e \sqrt{1 + 2p\sigma_H}$. That is, the shift map relates an incompressible QHF with Hall conductivity σ_H to a *putative* QHF with Hall conductivity σ'_H given by (37) by mapping the fusion ring \mathcal{A} of the chiral algebra $\mathcal{A} = \mathcal{C} \otimes \hat{u}(1)$ to a fusion ring \mathcal{A}' of the same chiral algebra as follows:

$$\mathcal{A} \ni \lambda = (\pi, r) \mapsto \lambda' = (\pi, r' = r \sqrt{1 + 2p\sigma_H}) \in \mathcal{A}' \quad (38)$$

Usually, \mathcal{A}' arises from the image of \mathcal{A} under the map introduced in (38) by adding further fractionally charged representations (keeping the set of representations of \mathcal{C} fixed). The only restriction comes from the requirement that the multi-electron fields are relatively local with respect to the fields corresponding to the new representations.

3.2. Remarks on Modular Invariance and Covariance

We wish to comment on the role that the modular group $SL(2, \mathbf{Z})$ may play in the analysis of incompressible QHF's.

For any TFT, one can consider the link in Fig. 4 in the three-sphere for any pair of representations λ and μ and compute the corresponding link invariant $S_{\lambda\mu}$, which is a complex number. Topological invariance implies that the matrix S with matrix elements $S_{\lambda\mu}$ is symmetric. Experience shows that requiring this matrix to be invertible ensures *completeness* of the theory, i.e., it ensures that one has included all types of static pointlike sources.

Let us assume that the fractional part of the conformal weights in each superselection sector for \mathcal{A} are constant. This assumption allows us to define a diagonal unitary matrix T by

$$T_{\lambda\mu} = \delta_{\lambda\mu} \exp(2\pi i(\Delta_\mu - c/24)) \quad (39)$$

Surgery operations for links in three-manifolds can be used to show that the matrices S and T generate a unitary representation of the modular group. If the fractional part of the conformal weight Δ is not constant, but the fractional part of $N\Delta$ is constant for all states in a given superselection sector, one still finds a representation of the subgroup of the modular group generated by S and T^N . It turns out that, in the description of incompressible QHF's in terms of CCFT's, it is quite natural to require covariance of the fusion ring under the subgroup of the modular group corresponding to $N=2$; see Appendix A.

We emphasize that the requirement of covariance of the fusion ring under subgroups of the modular group is formulated completely on the level of *chiral* CFT. It should not be confused with modular *invariance* of

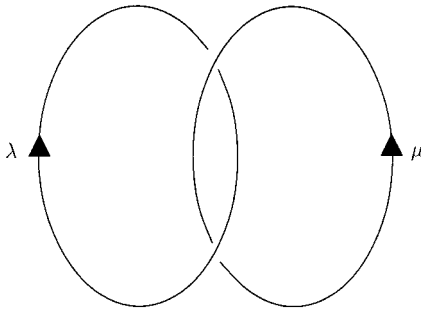


Fig. 4. The link whose link invariant in the three sphere S^3 equals $S_{\lambda\mu}$.

a torus partition function. The latter is a requirement in *full* CFT, which is a theory in which left movers and right movers have been combined. For the description of QHF's, chiral CFT is relevant; the consideration of partition functions that are invariant under subgroups of the modular group does not occur naturally.

3.3. Stability of the Incompressible Quantum Hall Fluid

Experimentally, only those incompressible QHF are accessible which are stable under small changes of the experimental control parameters, like, for example, the shape of the sample, the concentration of impurities or small inhomogeneities in the external magnetic field. This raises the question of how to assess the stability of an incompressible QHF described by a given TFT (and corresponding CCFT). In this paper, we shall not present an answer to this question based on an analysis of the microscopic quantum theory of incompressible QHF's. Instead, we propose some stability criteria extracted from the comparison of experimental data with theoretical predictions made in the framework of Quantum Hall Lattices.⁽²³⁾ These data indicate that an incompressible QHF is the more stable ...

(S1) ... the smaller the central charge $c_{\mathcal{A}} = 1 + c_{\mathcal{G}}$ is.

(S2) ... the smaller the conformal weights Δ_e of the electrons are.

(S3) ... the smaller the number ν_{frac} of representations with electric charge $0 \leq \mathbf{q} < 1$ is.

Concerning (S2) we remark that, experimentally, no incompressible QHF with $\sigma_H < \frac{1}{7}$ has been observed. The conformal weights of the electrons are bounded from below by $\Delta_e \geq 1/2\sigma_H$, a consequence of (34). This motivates the theoretical speculation that, for $\sigma_H < \frac{1}{7}$, or $\Delta_e > \frac{7}{2}$, the ground state of the system is a Wigner crystal, which is obviously not an incompressible state because of the existence of gapless modes (phonons). The criterion (S2) could then be interpreted as follows: the incompressible QHF is the more stable, the "farther remote" its groundstate is from such a crystalline ground state.

4. QUANTUM HALL LATTICES

In this section we review a description of incompressible QHF's in terms of a class of CCFT's whose chiral algebra is (an extension of) a $\hat{u}(1)^N$ -current algebra. This situation has been studied extensively, and has led to a partial classification of incompressible QHF in terms of Quantum Hall Lattices (QHL); see ref. 23.

Representations of a $\hat{u}(1)^N$ -current algebra are labelled by points \mathbf{r} in euclidean \mathbf{R}^N . The conformal weights are $\Delta_{\mathbf{r}} = \mathbf{r}^2/2$, and the corresponding primary fields are Wick-ordered exponentials of the form

$$\Psi_{\mathbf{r}} =: e^{i \sum_{j=0}^N r^j \phi_j} : \quad (40)$$

The real numbers $(r^i)_{i=1 \dots N}$ are the components of \mathbf{r} with respect to an orthonormal basis; the ϕ_i are massless chiral free fields with $\partial \phi_i = j_i$, where the j_i are the currents that generate the $\hat{u}(1)^N$ -current algebra with anomalous commutators

$$[j_i, m, j_j, n] = \delta_{m+n, 0} \delta_{ij} \quad (41)$$

The fusion product of representations is simply $\mathbf{r} * \mathbf{r}' = \mathbf{r} + \mathbf{r}'$.

In ref. 23, incompressible QHF's are described in terms of $\hat{u}(1)^N$ -current algebras. It is shown that:

(a) There is an odd integral lattice Γ in euclidean \mathbf{R}^N , with scalar product denoted by $\langle \cdot, \cdot \rangle$ such that the set of physically realized representations is a lattice Γ_{phys} with

$$\Gamma \subseteq \Gamma_{\text{phys}} \subseteq \Gamma^* \quad (42)$$

Here γ^* denotes the dual lattice of Γ .

(b) The electric current is $j = \sum_i Q^i j_i$, where $(Q^i)_{i=1 \dots N}$ are shown to be the components with respect to the chosen orthonormal basis of an *odd, primitive* vector $\mathbf{Q} \in \Gamma^*$. "Odd" means that for any vector $\mathbf{r} \in \Gamma$ we have $\langle \mathbf{Q}, \mathbf{r} \rangle = \langle \mathbf{r}, \mathbf{r} \rangle \pmod{2}$; "primitive" means that if it is joined to the origin by a line segment the latter does not contain any other point in Γ^* .

A CCFT with a $\hat{u}(1)^N$ -current algebra as chiral algebra and describing an incompressible QHF is therefore determined by a pair (Γ, \mathbf{Q}) with the properties described in (a) and (b). The Hall conductivity is given by

$$\sigma_H = \langle \mathbf{Q}, \mathbf{Q} \rangle \quad (43)$$

In the following we show that these data are equivalent to a Quantum Hall CCFT, (\mathcal{A}, A) , that fulfills the consistency conditions of Section 3.

(C1) The set $\{j = \sum_i K^i j_i \in \hat{u}(1)^N \mid \langle \mathbf{Q}, \mathbf{K} \rangle = 0\}$ generates an electrically neutral $\hat{u}(1)^{N-1}$ -current algebra. To obtain a rational theory in the electrically neutral sector, we pass to a simple current extension of $\hat{u}(1)^{N-1}$

with uncharged fields of integer conformal weight. This amounts to defining \mathcal{C} as the chiral algebra generated by

$$\left\{ j = \sum_i K^i j_i \in \hat{u}(1)^N \mid \langle \mathbf{Q}, \mathbf{K} \rangle = 0 \right\} \cup \left\{ : e^{i \sum_i r^i \phi_i} : \mid \mathbf{r} \in Q^\perp \right\} \quad (44)$$

where

$$Q^\perp = \{ \mathbf{r} \in \Gamma \mid \langle \mathbf{Q}, \mathbf{r} \rangle = 0 \} \quad (45)$$

Note that, by property (b), $\langle \mathbf{r}, \mathbf{r} \rangle$ is *even*, for every $\mathbf{r} \in Q^\perp$. The unitary representations of such \mathcal{C} are labelled by points in

$$\Pi = (Q^\perp)^*/Q^\perp \quad (46)$$

with

$$(Q^\perp)^* = \{ \mathbf{r}^\perp \mid \langle \mathbf{Q}, \mathbf{r}^\perp \rangle = 0; \langle \mathbf{n}, \mathbf{r}^\perp \rangle = 0 \pmod{1} \forall \mathbf{n} \in Q^\perp \} \quad (47)$$

(C2) The set of electron representations can be identified with

$$A_e = \{ \mathbf{r}_e \in \Gamma \mid \langle \mathbf{Q}, \mathbf{r}_e \rangle = 1 \} / Q^\perp \quad (48)$$

Note that the number of points in A_e is usually larger than 1. This means that, in general, there are several species of electrons distinguished from each other by some “internal quantum numbers”. The set of multi-electron representations is then given by

$$A_m = \{ \mathbf{r}_m \in \Gamma \mid \langle \mathbf{Q}, \mathbf{r}_m \rangle = j, j \in \mathbf{N}, j \geq 1 \} / Q^\perp \quad (49)$$

Multi-electron representation spaces are direct sums of countably many representation spaces of the $\hat{u}(1)^N$ -current algebra:

$$\mathcal{H}_m = \bigoplus_{\mathbf{s} \in Q^\perp} \mathcal{H}_{\mathbf{r}_m + \mathbf{s}} \quad (50)$$

The electric charge of such representations is a positive integer, $\mathbf{q}_m = \langle \mathbf{Q}, \mathbf{r}_m \rangle$, and is 1 for electrons. The set A_m can be obtained by multiple fusion of representations in A_e .

(C3) The conformal weights of multi-electron representations are

$$\Delta_m = \min_{\mathbf{s} \in Q^\perp} \{ \Delta_{\mathbf{r}_m + \mathbf{s}} \} \quad (51)$$

The charge and statistics connection is fulfilled, as follows from the definition of oddness of \mathbf{Q} :

$$2A_m \equiv \langle \mathbf{r}_m, \mathbf{r}_m \rangle = \langle \mathbf{Q}, \mathbf{r}_m \rangle = \mathbf{q}_m \pmod{2} \quad (52)$$

(C4) The complete set of representations is

$$A = \Gamma_{\text{phys}}/Q^\perp \quad (53)$$

and each representation is a direct sum of countably many representations of the $\widehat{u}(1)^N$ -current algebra:

$$\mathcal{H}_\lambda = \bigoplus_{\mathbf{s} \in Q^\perp} \mathcal{H}_{\mathbf{r}_\lambda + \mathbf{s}} \quad (54)$$

Again, the electric charge of such representations is given by $\mathbf{q}_\lambda = \langle \mathbf{Q}, \mathbf{r}_\lambda \rangle$. The fusion product of such representations corresponds to addition mod Q^\perp . The relative locality condition for the multi-electron fields reads

$$A_e + A_\lambda - A_{e * \lambda} = -\langle \mathbf{r}_e, \mathbf{r}_\lambda \rangle = 0 \pmod{1} \quad (55)$$

for all $e \in A_e$ and $\lambda \in A$. It is fulfilled, since $\mathbf{r}_e \in \Gamma$ and $\mathbf{r}_\lambda \in \Gamma^*$.

(C5) The electrically neutral algebra \mathcal{C} defined through (44) contains an $\widehat{su}(2)$ -current algebra at level 1 iff the lattice Q^\perp factorizes into sublattices and one of these factors is the root lattice of $su(2)$. This can occur in the case of “maximally symmetric QHL’s:” A maximally symmetric QHL is denoted by $(L |^\alpha \mathbf{g})$: \mathbf{g} is a semisimple Lie algebra, α a minimal weight thereof and L is an odd positive integer, with $L > (\alpha, \alpha)$; the QHL is generated by the root lattice of \mathbf{g} and by $\mathbf{r}_e = \alpha + (L - (\alpha, \alpha)) \mathbf{Q}$, where \mathbf{Q} is the charge vector perpendicular to the root lattice of \mathbf{g} . There is a unique electron representation, e , which corresponds to \mathbf{r}_e . If \mathbf{g} has an $su(2)$ -factor then the electron must have $\widehat{su}(2)$ -spin $\frac{1}{2}$, since α is minimal. The charge and spin connection in such a case is seen to hold for the electron representation; for multi-electron representations, that connection follows from the additivity of the electric charge under fusion and from the additivity mod 2 of $\widehat{su}(2)$ -spin.

The parameters that enter in the stability criteria are connected to lattice invariants:

(S1) The central charge of the Quantum Hall CCFT is the rank, N , of the lattice.

(S2) The largest conformal weight of the electrons is bounded from above by a lattice invariant called l_{max} ; see ref. 23 for more detailed explanations.

(S3) The number of representations with fractional electric charge is bounded from above by the discriminant, $|\Gamma^*/\Gamma|$, of the lattice.

5. CONSTRUCTION OF CHIRAL CONFORMAL THEORIES DESCRIBING INCOMPRESSIBLE QUANTUM HALL FLUIDS

The goal of this section is to describe a simple algorithm for the construction of Quantum Hall CCFT's.

5.1. Explicit Construction with Electrons as Simple Currents

The main assumption underlying our algorithm is that electrons are *simple currents*; for a review on simple currents see ref. 47.

An irreducible representation of a chiral algebra is called a simple current if its fusion with any other irreducible representation yields exactly one irreducible representation. Equivalently, a simple current e is characterized by the fusion rule

$$e * \bar{e} = \omega \quad (56)$$

where \bar{e} is the representation conjugate to e . Thus our assumption simply means that when a hole is filled with a corresponding electron, the state of the system is the vacuum state. When dealing with simple currents, a useful concept is that of *monodromy charge*. The monodromy charge (or simply the monodromy) of a representation λ with respect to a simple current e is given by

$$Q_e(\lambda) = \Delta_e + \Delta_\lambda - \Delta_{e * \lambda} \bmod 1 \quad (57)$$

If e, e_1, e_2 are simple currents, $\lambda, \lambda_1, \lambda_2$ arbitrary representations, we have that

$$Q_{e_1}(e_2) = Q_{e_2}(e_1) \quad (58)$$

$$Q_e(\lambda_1 * \lambda_2) = Q_e(\lambda_1) + Q_e(\lambda_2) \bmod 1 \quad (59)$$

$$Q_{e_1 * e_2}(\lambda) = Q_{e_1}(\lambda) + Q_{e_2}(\lambda) \bmod 1 \quad (60)$$

The set of simple currents of a theory, endowed with the fusion product $*$, has the structure of an abelian group, with unit corresponding to the vacuum representation ω . We note that a simple current is relatively local iff all representations occurring in the theory have vanishing monodromy with respect to the simple current.

The assumption that electron fields are described by simple currents drastically simplifies the analysis of the consistency conditions (C3) and (C4) of Section 3.1. In the following, we show how to construct a Quantum Hall CCFT satisfying conditions (C1) through (C5) and (S1) through (S3) (see Section 3.1 and 3.3) with electron fields given by simple currents.

(i) The electron representations, A_e , are of the form $e_a = (\varepsilon_a, 1/\sqrt{\sigma_H})$, where ε_a are simple currents of the \mathcal{C} -theory, with the property that

- the charge-statistics connection (C1) is satisfied for each electron, i.e.,

$$\frac{1}{\sigma_H} + 2A_{\varepsilon_a} = 1 \pmod{2}, \quad a = 1, \dots, v_e \quad (61)$$

where v_e is the number of distinct species of electrons; and

- the relative locality condition is satisfied for each pair of electrons,⁶ i.e.,

$$Q_{e_a}(e_b) = -\frac{1}{\sigma_H} + Q_{\varepsilon_a}(\varepsilon_b) = 0 \pmod{1}, \quad a, b = 1, \dots, v_e \quad (62)$$

(ii) As a consequence of (i), the multi-electron representations, A_m , obtained by multiple fusion of electron representations are again simple currents. The relative locality condition (C4), $Q_{m_1}(m_2) = 0$, for $m_1, m_2 \in A_m$, is fulfilled, as follows from (58), (59), and (60). By induction, it is possible to prove that the charge-statistics connection (C3) is fulfilled, because from $A_{m_1} = \frac{1}{2}\mathbf{q}_{m_1} \pmod{1}$, and $Q_{m_1}(m_2) = 0$, it follows that $A_{m_1 * m_2} = \frac{1}{2}\mathbf{q}_{m_1 * m_2} \pmod{1}$.

(iii) Further representations, $\lambda = (\pi, r)$, have to fulfill the relative locality condition (C4). This amounts to requiring the vanishing of the monodromy charges with respect to the electrons, i.e.,

$$Q_{e_a}(\lambda) = -\frac{\mathbf{q}_r}{\sigma_H} + Q_{\varepsilon_a}(\pi) = 0 \pmod{1} \quad a = 1, \dots, v_e \quad (63)$$

The vanishing of the monodromy charge with respect to multi-electron representations trivially follows from (60). If \mathcal{A} is defined to consist of *all* representations λ satisfying (63) then \mathcal{A} is closed under fusion. From (59)

⁶ If there is only one electron, then (62) follows directly from (61). This can be proven using the identities $2A_e = r(N-1)/N$ and $Q_e(\varepsilon) = r/N$, where N is the order of ε , and r is an integer defined mod N .

it indeed follows that fusion of two representations with vanishing monodromy charge with respect to a simple current gives representations with vanishing monodromy charge with respect to the (same) simple current.

Equation (63) defines, for each representation π of \mathcal{C} , a discrete set of charges

$$\mathbf{q} = \sigma_H Q_e(\pi) + \sigma_H k, \quad k \in \mathbf{Z} \quad (64)$$

or, equivalently, a discrete set of (labels of) $\delta\hat{u}(1)$ -representations

$$r = \sqrt{\sigma_H} Q_e(\pi) + \sqrt{\sigma_H} k, \quad k \in \mathbf{Z} \quad (65)$$

Since σ_H is a rational number and \mathcal{C} is rational, it is possible to find an even integer, \mathcal{N} , such that all $\hat{u}(1)$ -labels that occur in \mathcal{A} are of the form $r = l/\sqrt{\mathcal{N}}$, for a suitable integer l . The set of physically realizable representations is contained in

$$\mathcal{A} \subseteq \Pi \times \mathbf{Z} \left[\frac{1}{\sqrt{\mathcal{N}}} \right] \quad (66)$$

as required by condition (63).

In general, a large amount of mathematical information can be obtained from the data of a CCFT. However, only a small part of that information can be related to *experimentally accessible* physical quantities of an incompressible QHF. Among such physical quantities is the *minimal electric charge of a quasi-particle* of the QHF.^(48,49) We can compute this quantity very easily as follows.

For simplicity, we assume that there is only one species of electrons present in the theory. The neutral factor, ε , of the electron field is a simple current of the \mathcal{C} theory. According to ref. 47, the conformal weight of any simple current ε is related to its order, $\text{ord}(\varepsilon)$, by an equation of the form:

$$2A_\varepsilon = \frac{\text{ord}(\varepsilon) - 1}{\text{ord}(\varepsilon)} r_\varepsilon \bmod \mathbf{Z} \quad (67)$$

where r_ε is an integer-valued quantity associated with the simple current, defined modulo $\text{ord}(\varepsilon)$. From equation (61), writing $\sigma_H = n_H/d_H$ for relatively prime integers n_H and d_H , we conclude that $\text{ord}(\varepsilon)$ must be a multiple of n_H ,

$$\text{ord}(\varepsilon) = \ell n_H \quad (68)$$

Here, ℓ must be a divisor of the quantity r_ε . The monodromy charge of any other representation, π , with respect to ε is of the form $r_\pi/\text{ord}(\varepsilon)$,⁽⁴⁷⁾ for some integer r_π . Combining this fact with (68) and with (63), we conclude that the smallest possible electric charge is

$$\mathbf{q}_{\min} = \frac{1}{\ell d_H} \quad (69)$$

We cannot, of course, assert that this smallest possible charge can be realized experimentally. An equation similar to (69) has been derived for Quantum Hall lattices in ref. 23, where the analogue of ℓ was called *charge parameter*.

5.2. Coset and Orbifold Construction of Quantum Hall CCFT's

Recently, the application of the *coset construction* to the quantum Hall effect has attracted some attention,⁽²⁹⁾ because it provides a possibility to relate the maximally symmetric QHL ($1 |^\alpha su(2) \oplus su(2)$), with $\sigma_H = \frac{1}{2}$, $c_{\mathcal{A}} = 3$, to a $\text{Vir}_1 \times \hat{u}(1)$ -theory (Vir_1 being the chiral algebra of the Ising model), with $\sigma_H = \frac{1}{2}$, $c_{\mathcal{A}} = \frac{3}{2}$.

The relevance of the coset construction (see Appendix B) for our framework is twofold. First, the coset construction allows one to construct a new class of theories, starting from WZW-models, always lowering the central charge. In view of stability criterion (S1), the coset theories provide good candidates for the theory corresponding to the electrically neutral chiral algebra \mathcal{C} . Second, if there are gauge symmetries present in the theory corresponding to the incompressible QHF, which may be expected on physical grounds in a particular situation, the coset construction is a method to implement the gauge reduction: coset CCFT can be viewed as WZW-models in which a subgroup is gauged.

One might want to also consider *orbifold theories* as candidates for the description of incompressible QHF's. In view of the stability conditions of Section 3.2 they do not appear to be particularly promising candidates, though: in contrast to the coset construction (see Appendix B), the orbifold construction does *not* lower the Virasoro central charge. Moreover, the orbifold construction requires additional fields, so-called twist fields, for any primary field that is symmetric under the action of the orbifold group. (For a discussion in the case when the orbifold group is \mathbf{Z}_2 , see, e.g., ref. 50). Unless there are plenty of primary fields that are not symmetric under the action of the orbifold group, the orbifold theory will therefore have more primary fields than the original theory. Thus, in general, the orbifold theory can be expected to be less stable than the original theory.

6. EXAMPLES OF QUANTUM HALL FLUIDS WITH

$$\sigma_H = \frac{1}{2} (e^2/h), \frac{3}{5} (e^2/h), (e^2/h), \dots$$

In this section, we illustrate the construction proposed in Section 5 by some simple but important examples. Candidates for the electrically neutral theory \mathcal{C} are the Virasoro minimal models, simple current extensions thereof, and low-rank, low-level WZW-models. (WZW models at level 1 are encountered in connection with maximally symmetric QHL, as discussed in Section 4.) These three classes of examples have small central charge, simple currents with small conformal weight defining electron fields, and they have a rather small number of unitary representations. These are favourable features, in view of the stability criteria (S1), (S2) and (S3) of Section 3.3.

6.1. Virasoro Minimal Models

The Virasoro minimal models are labelled by a strictly positive integer k . They can be obtained from the coset construction,⁽⁵¹⁾

$$\text{Vir}_k \cong \frac{(\widehat{su}(2))_k \times (\widehat{su}(2))_1}{(\widehat{su}(2))_{k+1}}, \quad c_k = 1 - \frac{6}{(k+2)(k+3)} \quad (70)$$

Each model has exactly one simple current ε of order two ($\varepsilon^{*2} = \omega$), with vanishing self-monodromy $Q_\varepsilon(\varepsilon) = 0$. Its conformal weight is given by

$$\begin{aligned} \Delta_\varepsilon &= \frac{k(k+1)}{4} = \frac{k}{4} \bmod 1, & \text{for } k \text{ even} \\ \Delta_\varepsilon &= \frac{k(k+1)}{4} = -\frac{k+1}{4} \bmod 1, & \text{for } k \text{ odd} \end{aligned} \quad (71)$$

If we use these simple currents to construct the electron representation, then the charge-statistics connection (61) and relative locality (62) can be fulfilled if the values of the Hall conductivity are restricted to

$$\begin{aligned} k = 1, 2 \bmod 4 &\Rightarrow \frac{1}{\sigma_H} = 2, 4, \dots \\ k = 3, 4 \bmod 4 &\Rightarrow \frac{1}{\sigma_H} = 1, 3, \dots \end{aligned} \quad (72)$$

These series of values of σ_H^{-1} can be obtained by applying the shift map to the theories with $\sigma_H = 1/2, 1$, respectively. We first consider the simplest example, $\mathcal{C} = \text{Vir}_1$ (Ising-model). The largest possible value of the Hall

Table I. Relevant Features of the Vir_1 -Model: Conformal Weights of the Representations, the Action of the Simple Current on the Representations and the Monodromy Charges of the Representations

	Δ	$\varepsilon * (\cdot)$	$Q_\varepsilon(\cdot)$
Ω	0	ε	0
ε	$\frac{1}{2}$	ω	0
σ	$\frac{1}{16}$	σ	$\frac{1}{2}$

conductivity is $\delta\sigma_H = 1/2$, which is an interesting Hall plateau, since it is one of the few observed plateaux with an *even* denominator. The model has one nontrivial simple current, ε . The relevant features of the model are summarized in Table I. Applying the algorithm of Section 5, we find a set of representations, \mathcal{A} , which is represented in Fig. 5. In this example, the Hall conductivity of the theory based on $(\widehat{su}(2))_1 \times (\widehat{su}(2))_1$ which is described by a quantum Hall lattice, and the Hall conductivity of the theory based on the coset $(\widehat{su}(2))_1 \times (\widehat{su}(2))_1 / (\widehat{su}(2))_2$ are identical (modulo shift map). For a proof, see Appendix B.

The same procedure can be applied to the higher- k minimal models. The main features of some of these theories are represented in Table II. We note that Eq. (71) together with the requirement $\Delta_e \leq 7/2$ (see the remark concerning the stability criterion (S2) in Section 3.3) restricts the number of minimal models that can be expected to describe a stable QHF by $k \leq 3$.

6.2. Simple Current Extensions of Virasoro Minimal Models

The (unique) simple current of a Vir_k -model has integer conformal weight for $k = 3, 4 \pmod{4}$, in which case the model allows for an extension

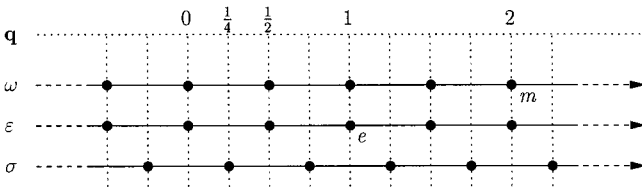


Fig. 5. Physically realized representations for $\mathcal{C} = \text{Vir}_1$; e marks the electron representation, m a multi-electron representation and \bullet the representations that fulfill (63) and hence appear in the theory.

Table II. Features of Theories Constructed with Virasoro-Minimal Models in the Neutral Sector \mathcal{C} . We Indicate the Central Charge $c = c_{\mathcal{A}}$, the Electron Conformal Weight Δ_e and the Number of Fractionally Charged Representations v_{frac} . In Addition, We Give the Integer \mathcal{N} that Can Be Used to Construct Covariant Characters Under $\Gamma_2(\mathcal{S})$ (see Appendix A)

Theory		Stability			
\mathcal{C}	σ_H	c	Δ_e	v_{frac}	\mathcal{N}
Vir ₁	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	6	8
Vir ₂	$\frac{1}{2}$	$\frac{17}{10}$	$\frac{5}{2}$	12	8
Vir ₃	1	$\frac{9}{5}$	$\frac{7}{2}$	10	4

of the chiral algebra by the primary field corresponding to the simple current. It is then possible that in the set of representations of the *extended* algebra some new simple currents are encountered.

We first consider the easiest example, with $k = 3$. The simple current of Vir₃ has conformal weight $\Delta = 3$, and the extension is known as the W_3 -minimal model or three-states Potts-model. Its simple current group is $\{\omega, \varepsilon, \varepsilon'\} \cong \mathbf{Z}_3$. The relevant features of the model are given in Table III. We may use a simple current, ε , say, to construct the electron representation, but *not* both simple currents ε and ε' , because this would violate the relative locality requirement. Equations (61) (charge-statistics connection) and (62) (relative locality) can be fulfilled if the Hall conductivity is given by

$$\frac{1}{\sigma_H} = \frac{5}{4} + 2l \quad (73)$$

Table III. Relevant Features of the W_3 -Model: Conformal Weights of the Representations, the Action of the Simple Current on the Representations and the Monodromy Charges of the Representations

	Δ	$\varepsilon *$	$Q_\varepsilon(\cdot)$	$\varepsilon' *$	$Q_{\varepsilon'}(\cdot)$
Ω	0	ε	0	ε'	0
ε	$\frac{2}{3}$	ε'	$\frac{2}{3}$	ω	$\frac{1}{3}$
ε'	$\frac{2}{3}$	ω	$\frac{1}{3}$	ε	$\frac{2}{3}$
α	$\frac{2}{5}$	β	0	γ	0
β	$\frac{1}{15}$	γ	$\frac{2}{3}$	α	$\frac{1}{3}$
γ	$\frac{1}{15}$	α	$\frac{1}{3}$	β	$\frac{2}{3}$

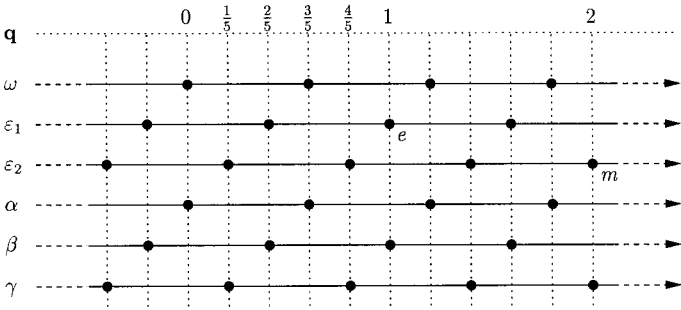


Fig. 6. Representations for $\mathcal{C} = W_3$; e marks the electron representation, m a multi-electron representation and \bullet the representations that fulfill (63) and hence can appear in the theory.

with l a positive integer. Applying the construction of Section 5 we find a set of representations, \mathcal{A} , represented in Fig. 6.

Let us finally consider the case $k=4$. The simple current has conformal weight $\Delta=5$, and the extension is known as the W_5 -minimal model. It has nine unitary representations and a \mathbf{Z}_3 -group $\{\omega, \varepsilon, \varepsilon'\}$ of simple currents, with $\Delta_\varepsilon = \Delta_{\varepsilon'} = \frac{4}{3}$. The construction of Section 5 can be applied using ε to construct the electron representation; the values of the Hall conductivity are restricted to $1/\sigma_H = \frac{1}{3} + 2l$, with l a positive integer. The main features of the above theories are represented in Table IV.

APPENDIX A. MODULAR COVARIANCE UNDER $\Gamma_2(\mathcal{S})$

Suppose, for simplicity, that there is only one species of electrons. Then we show that the linear space spanned by certain characters of the theories constructed according to the algorithm of Section 5 carries a

Table IV. Features of Theories Constructed with W_3 - and W_5 -Models in the Neutral Sector \mathcal{C} . We Indicate the Central Charge $c = c_{\mathcal{A}}$, the Electron Conformal Weight Δ_e and the Number of Fractionally Charged Representations v_{frac} . In Addition, We Give the Integer \mathcal{N} that Can Be Used to Construct Covariant Characters Under $\Gamma_2(\mathcal{S})$ (see Appendix A)

Theory		Stability			
\mathcal{C}	σ_H	c	Δ_e	v_{frac}	\mathcal{N}
W_3	$\frac{3}{5}$	$\frac{9}{5}$	$\frac{3}{2}$	10	60
W_5	3	$\frac{13}{7}$	$\frac{3}{2}$	3	12
	$\frac{3}{7}$	$\frac{13}{7}$	$\frac{3}{2}$	21	84

representation of a subgroup of the modular group $SL(2, \mathbf{Z})$, usually denoted by $\Gamma_2(S)$. This group is generated by the operators S and T^2 introduced in Section 3.2.

The proof of our claim goes as follows.

(i) The even integer \mathcal{N} can be chosen such that, for a suitably chosen positive integer n , the n -fold fusion of the electron representation is given by

$$e^{*n} = \left(\varepsilon^{*n}, \frac{n}{\sqrt{\sigma_H}} \right) = (\omega, \sqrt{\mathcal{N}}) \quad (74)$$

where the electron representation is given by $e = (\varepsilon, d/\sqrt{N})$, with d a positive integer. We have that $\mathcal{N} = nd$.

(ii) The representations of $\hat{u}(1)$ -current algebra labelled by the set $\{r = l/\sqrt{\mathcal{N}} \mid l \in \mathbf{Z}\}$ can be grouped into \mathcal{N} (reducible) $\hat{u}(1)$ -representations by building the direct sum of representation spaces

$$\bar{\mathcal{H}}_k = \bigoplus_{l \in \mathbf{Z}} \mathcal{H}_{((k + \mathcal{N}l)/\sqrt{\mathcal{N}})}, \quad 0 \leq k \leq \mathcal{N} - 1 \quad (75)$$

The linear space spanned by the corresponding characters

$$\bar{\chi}_k(\tau) = \text{tr}_{\bar{\mathcal{H}}_k} [e^{2\pi i \tau (L_0 - (c/24))}] \quad (76)$$

carries a representation of the modular group given by the matrices

$$\bar{T}_{kl} = \delta_{kl} e^{2\pi i \Delta_k}, \quad \Delta_k = \frac{k^2}{2\mathcal{N}} \quad (77)$$

and

$$\bar{S}_{kl} = \frac{1}{\sqrt{\mathcal{N}}} e^{-2\pi i (kl/\mathcal{N})} \quad (78)$$

(iii) The space of characters of the \mathcal{C} -theory, defined by

$$\chi_\pi(\tau) = \text{tr}_{\mathcal{H}_\pi} [e^{2\pi i \tau (L_0 - c/24)}] \quad (79)$$

carries a representation of the modular group, given by the diagonal matrix $T_{\pi\pi'} = \delta_{\pi\pi'} e^{2\pi i (A_\pi - c/24)}$ and some unitary, symmetric matrix $S_{\pi\pi'}$.

(iv) The (reducible) representations of the algebra $\mathcal{C} \times \hat{u}(1)$ on the spaces

$$\mathcal{H}_{(\pi, k)} = \mathcal{H}_\pi \otimes \bar{\mathcal{H}}_k \tag{80}$$

have characters

$$\bar{\chi}_{(\pi, k)}(\tau) = \chi_\pi(\tau) \cdot \bar{\chi}_k(\tau) \tag{81}$$

These characters are considered as linearly independent and span a finite dimensional vector space, which we denote as $\text{Char}_{(\Pi, \mathcal{N})}$. It carries a representation of the modular group, given by matrices $\bar{T}_{(\pi, k)(\pi', k')} = T_{\pi\pi'} \cdot \bar{T}_{kk'}$ and $\bar{S}_{(\pi, k)(\pi', k')} = S_{\pi\pi'} \cdot \bar{S}_{kk'}$.

(v) The set of representations $\{(\pi, k) \mid \pi \in \Pi, k = 0, \dots, \mathcal{N} - 1\}$ decomposes into orbits under the action of the abelian group of simple currents $\{\omega, e, \dots, e^{*(n-1)}\}$, the action being given by

$$e * (\pi, k) = (\varepsilon * \pi, k + d \bmod \mathcal{N}) \tag{82}$$

The orbits have all the same length, n .

(vi) Let us define characters of the orbits by setting

$$\hat{\chi}_{\lfloor \pi, k \rfloor} = \bar{\chi}_{(\pi, k)} + \bar{\chi}_{e * (\pi, k)} + \dots + \bar{\chi}_{e^{*n} * (\pi, k)}, \quad 0 \leq k \leq d - 1 \tag{83}$$

Consider only those characters $\hat{\chi}$, for which (π, k, \mathcal{N}) is in the set of physically realizable representations \mathcal{A} . They are linearly independent and span a subspace of $\text{Char}_{(\Pi, \mathcal{N})}$, which we denote as $\text{Char}_{\mathcal{A}}$. These characters involve exactly those representations that appear in the Quantum Hall CCFT.

(vii) We now prove that the subspace $\text{Char}_{\mathcal{A}}$ of $\text{Char}_{(\Pi, \mathcal{N})}$ is invariant under the action of T^2 and S . For T^2 , this follows from the fact that, in an orbit, there appear only representations whose conformal weights differ by half-integers. For, we have that Δ_e is half-integer and that $Q_e(\lambda) = 0$, and therefore the difference $\Delta_{e * \lambda} - \Delta_\lambda = \Delta_e$ is a half-integer. Thus, we have that

$$(T^2 \hat{\chi}_{\lfloor \pi, k \rfloor})(\tau) = \hat{\chi}_{\lfloor \pi, k \rfloor}(\tau + 2) = e^{2\pi i \cdot (2\Delta_{(\pi, k)} - c/12)} \hat{\chi}_{\lfloor \pi, k \rfloor}(\tau) \tag{84}$$

For S , we reason as follows. Let \mathcal{A} be a finite set of representations of a chiral algebra for which there is a unitary and symmetric matrix S implementing the transformation $\tau \mapsto -1/\tau$ on the characters. Let λ, λ'

be representations and e a simple current in \mathcal{A} ; then we have that $S_{e * \lambda, \lambda'} = S_{\lambda, \lambda'} e^{2\pi i \cdot Q_e(\lambda')}$. A simple calculation shows that

$$(S\hat{\chi}_{\lfloor \pi, k \rfloor})(\tau) = \hat{\chi}_{\lfloor \pi, k \rfloor} \left(-\frac{1}{\tau} \right) = \sum_{(\pi', k')} n S_{(\pi, k)(\pi' k')} \hat{\chi}_{\lfloor \pi, k' \rfloor}(\tau) \tag{85}$$

Thus, we have proven that the space of characters $\text{Char}_{\mathcal{A}}$ carries a representation of the group $\Gamma_2(S)$, and that, in the canonical basis (83), the corresponding matrices read

$$\hat{T}_{\lfloor \pi, k \rfloor, \lfloor \pi', k' \rfloor}^2 = \delta_{\lfloor \pi, k \rfloor, \lfloor \pi', k' \rfloor} e^{2\pi i \cdot (2A_{(\pi, k)} - c/12)} \tag{86}$$

and

$$\hat{S}_{\lfloor \pi, k \rfloor, \lfloor \pi', k' \rfloor} = n \bar{S}_{(\pi, k), (\pi', k')} \tag{87}$$

Since \bar{S} is symmetric, the same is true for \hat{S} ; \hat{S} is also unitary, since it is the restriction of a unitary linear map on an invariant subspace.

APPENDIX B. COSETS

The coset construction^(27, 28) provides a powerful tool to construct (chiral) conformal field theories. In this appendix, we describe the main idea and sketch a few important features of this construction.

The starting point is a WZW theory, i.e., a theory based on non-abelian currents. The zero-modes of these currents form a Lie algebra \mathfrak{g} . Natural examples of particular interest for our purposes are provided by WZW theories based on simply laced Lie algebras at level one: they can also be described by a lattice theory based on the root lattice of the Lie algebra. The affine Sugawara construction provides a chiral stress energy tensor, $T(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2}$, whose Fourier modes L_n span a Virasoro algebra with a certain central charge c . (In the case of the lattice model, c equals the rank of the lattice.)

Next, one fixes a subalgebra, \mathfrak{g}' , of \mathfrak{g} . The affine Sugawara construction applied to \mathfrak{g}' yields another Virasoro algebra L'_n , with a different central charge $c' \leq c$. The crucial observation is that the operators

$$\dot{L}_n := L_n - L'_n \tag{88}$$

form a third Virasoro algebra, with central charge $\dot{c} = c - c'$ in the range $c > \dot{c} \geq 0$, which is the Virasoro algebra of the so-called coset theory. The coset construction therefore lowers the Virasoro central charge.

The conformal weights $\hat{\Delta}$ in the coset theory are given by differences of the conformal weights,

$$\hat{\Delta} = \Delta - \Delta' \bmod \mathbf{Z} \quad (89)$$

In particular, the currents in \mathfrak{g}' have zero conformal weight: they are “gauged.” Indeed, full coset CFT’s admit a description as gauged WZW theories.⁽⁵²⁾

As a consequence, the starting point for the construction of the state space of a coset theory are so-called branching spaces, i.e., the spaces of multiplicities of \mathfrak{g}' -representations in \mathfrak{g} -representations. The precise construction of the state space is actually quite subtle, and we refer the reader to refs. 53, 40 for details. In particular, despite numerous claims in the literature, the spaces of physical states are in general *not* the branching spaces, but suitable subspaces thereof.

We only mention that, in analogy to the \mathbf{Z}_2 symmetry of the Kac table, typically different branching spaces provide different representatives for one and the same physical state. This effect, which goes under the name of “field identification,” can be understood in almost all cases in terms of group theoretical selection rules. Additional subtleties occur if this field identification has so-called fixed points. In this case, branching spaces have to be split and one branching space contains states of different primary fields. The corresponding algorithm has been worked out for diagonal cosets in ref. 40.

As an illustration of the concepts involved in the coset construction, we provide a criterion for those cases in which the theory based on the coset describes a QHF with the same Hall conductivity as the original theory. If ε is a simple current of the WZW theory based on the Lie algebra \mathfrak{g} with conformal weight Δ_ε and if λ' is a simple current of the theory based on the subalgebra \mathfrak{g}' , with conformal weight $\Delta_{\lambda'}$, such that the branching space associated to ε and λ' is non trivial, then there is a simple current (ε, λ') of the coset theory with conformal weight

$$\Delta_{(\varepsilon, \lambda')} = \Delta_\varepsilon - \Delta_{\lambda'} \bmod \mathbf{Z} \quad (90)$$

In particular, if λ' is the vacuum representation of the \mathfrak{g}' theory, then $\Delta_{(\varepsilon, \lambda')} = \Delta_\varepsilon \bmod \mathbf{Z}$. Thus if ε is the electrically neutral part of a one-electron representation of the $\hat{u}(1) \otimes \mathfrak{g}$ -theory, and λ' is the vacuum representation of the \mathfrak{g}' theory, then (ε, λ') is the electrically neutral part of a one-electron representation of the coset theory. In this case, formula (61), i.e.,

$$\frac{1}{\sigma_H} + 2\Delta_\varepsilon = \frac{1}{\hat{\sigma}_H} + 2\Delta_{(\varepsilon, \lambda')} = 1 \bmod 2 \quad (91)$$

tells us that the original theory and the coset construction have the *same* Hall conductivity, modulo the shift map. In particular, this criterion can be used to identify possible coset constructions based on maximally symmetric quantum Hall lattices.

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